ON A COMMUTATIVE RING STRUCTURE IN QUANTUM MECHANICS

SHIGEKI MATSUTANI

Abstract. In this article, I propose a concept of the p-on which is modelled on the multi-photon absorptions in quantum optics. It provides a commutative ring structure in quantum mechanics. Using it, I will give an operator representation of the Riemann ζ function.

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1. Introduction

The integer appearing quantum mechanics basically comes from eigenvalue of an operator, which merely has the additive structure. On the other hand, number theory is a study of the integer as a commutative ring rather than an additive group. There prime numbers play the crucial roles whereas they basically have no meaning in an additive group. However number theory and quantum mechanics sometimes are connected [M, M1, M2, MO, V1, V2, VVZ]. I attempted to answer a question why quantum mechanics is connected with integer theory in [M2, MO]. This article is one of the attempts. I will explore a commutative ring structure in the harmonic oscillator.

In [BK, M2, MO], it was showed that the Gauss sum which is a number theoretic object plays the central roles in an interference phenomenon, the fractional Talbot phenomenon [WW]. As I investigated the algebraic structures behind the connection between wave physics and number theory, there are $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ [M2]. Though it is well-known, the generating relation of the Lie algebra of the abelian extension of $\mathrm{SL}(2,\mathbb{R})$, the Heisenberg group, is $[\frac{d}{dx}, x] = 1$ [LV] whereas the defining relation of $\mathrm{SL}(2,\mathbb{Z})$ is $pa + qb = 1$ of $\begin{pmatrix} p & q \\ -b & a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. These relations are essential in quantum mechanics and number theory respectively [RS, IR] and also of the connection in the classical optical phenomenon [M2].

On the other hand, the Heisenberg group and the interference phenomena are represented by the Fourier series [LV, RS]. The Fourier series is the representation space of an additive group or the translation group. The translation group plays crucial roles in

the interference phenomena and thus must be essential in the connection between number theory and wave physics.

In the computation of the (discrete) Fourier transformation, the algorithm of the fast Fourier transformation is well-known, which is based upon a commutative ring structure of the Fourier series [T]: For a composite integer $\ell = pq$ and $k \in \mathbb{R}$, we have

$$\exp(\sqrt{-1}kpq) = \left(\mathrm{e}^{\sqrt{-1}kp}\right)^q = \left(\mathrm{e}^{\sqrt{-1}kq}\right)^p. \tag{1.1}$$

This commutative ring structure is the key structure in the fast Fourier transformation. I consider that it also plays the crucial roles in the connection in the interference phenomenon [BK, MO, M2], though the property (1.1) comes from the primitive fact that the set of the integers naturally has a commutative ring structure. In other words, the Fourier series as the representation space of the additive group brings the commutative ring structure to the interference phenomenon and contributes to the connection between number theory and quantum mechanics.

Eigenvalue of the creation operator in the harmonic oscillator is given by non-negative integers which is merely given by an additive (semi-)group generated by 0 and +1. However even for the harmonic oscillator, we may have such a commutative ring structure based upon the primitive fact. Indeed, in quantum optics, the multi-photon absorptions are observed and play the important roles. Algebraic structure of two-photon absorptions was studied by Brif [B]. In this article, we introduce an operator p-on, which is modelled on p-photon absorptions, in order to introduce the commutative ring structure into quantum mechanics. Further we also define a quantum p-on operator.

Related to the harmonic oscillator in quantum statistical mechanics and field theory, the Riemann ζ function [Pa],

$$\zeta(s) = \sum_{n=1} \frac{1}{n^s},$$

naturally appears [IZ, C] as shown in the Appendix. In fact, the Planck's black body problem is related to $\zeta(4)$ [Pl] and the Casimir effect is to $\zeta(-3)$ [C].

The Riemann ζ function was studied by Bost and Connes [BC, CM] in the framework of non-commutative algebra, which corresponds to quantum statistical mechanics physically speaking. The Riemann ζ function has the Euler product expression [Pa],

$$\zeta(s) = \prod_{p:\text{prime number}} \frac{1}{1-p^s}, \tag{1.2}$$

which plays crucial roles in number theory. The prime number has special meanings in the expression. In the paper [BC], there appeared an operator whose eigenvalues are prime numbers.

One of the purposes of this article is to show its quantum version of the Euler product expression and a quantum mechanical meaning of (1.2) in the harmonic oscillator. In other words, in this article, I will show that even in harmonic oscillator whose eigenvalues are mere integers as an additive semi-group, there are expressions related to the Euler

product expression (1.2) if we handle p-on and quantum p-on. In Discussion I mention that a quantum Euler product expression of the Riemann ζ function might be related to the absolute derivation [KOW].

2. p-on

The harmonic oscillator in quantum mechanics provides the integer as its eigenvalue of the eigenstates [Di]. The harmonic oscillator is given by the Hamiltonian,

$$H = \frac{1}{2}(a^\dagger a + a a^\dagger),$$

using the creation operator $a^\dagger$ and the annihilation operator a which satisfy the canonical communication relations,

$$[a, a^\dagger] = 1, \quad [a^\dagger, a^\dagger] = [a, a] = 0. \tag{2.1}$$

Let the vacuum states be denoted by $|0\rangle$ and $\langle 0|$, *i.e.*, $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$. Let the infinite dimensional $\mathbb{C}$ vector space generated by $a^\dagger$ (a) be denoted by $\mathfrak{a}^+$ $(\mathfrak{a}^-)$, *i.e.*, $\mathfrak{a}^+ := \mathbb{C}[[a^\dagger]]|0\rangle$ $(\mathfrak{a}^- := \langle 0|\mathbb{C}[[a]])$, where $\mathbb{C}[[a^\dagger]]$ $(\mathbb{C}[[a]])$ is a commutative formal expansion algebra of $a^\dagger$ (a), *i.e.*, $f = \sum_{n=0}^\infty c_n {a^\dagger}^n$, $c_n \in \mathbb{C}$. The number state in $\mathfrak{a}^+$ and $\mathfrak{a}^-$ given by

$$\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle = |n\rangle, \quad \langle 0|\frac{1}{\sqrt{n!}}a^n = \langle n|,$$

which satisfies the orthonormal relation

$$\langle m|n\rangle = \delta_{n,m}.$$

Thus a subspace of $\mathcal{H} := (\mathfrak{a}^+, \mathfrak{a}^-, \langle|\rangle)$ becomes the Hilbert space. The number operator $\hat{n} := a^\dagger a$ picks out an integer n as its eigenvalue,

$$a^\dagger a|n\rangle = n|n\rangle.$$

These $a^\dagger$, a and $\hat{n}$ obey the relations

$$[\hat{n}, a^\dagger] = a^\dagger, \quad [\hat{n}, a] = -a. \tag{2.2}$$

In number theory, the set of integers is studied as a commutative ring rather than an discrete additive group. The eigenvalue of the harmonic oscillator is a mere additive semigroup because $a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle$ or $a^\dagger$ generates $+1$ action on the state $|n\rangle$.

On the other hand in quantum optics, multi-photon absorption, such as two-photon absorption, is known as an important phenomenon [B, L]. I show that this phenomenon brings a commutative ring structure into the harmonic oscillator.

The two-photon absorption occurs by the composite operator $a_2^\dagger := (a^\dagger)^2$ such that $a_2^\dagger|n\rangle = \sqrt{(n+2)(n+1)}|n+2\rangle$. Brif investigated the quantum system governed by

these composite operator [B]. He studied the Lie algebra given by the relations among $(\hat n, a_2^\dagger, a_2 := a^2, a^\dagger, a, 1)$. Besides (2.1) and (2.2), they obey

$$[a_2, a_2^\dagger] = 4\hat n + 2, \quad [\hat n, a_2^\dagger] = 2a_2^\dagger, \quad [\hat n, a_2] = -2a_2, \quad [a_2, a_2^\dagger] = 4\hat n + 2. \tag{2.3}$$

Brif investigated its representation space precisely. Further we note the relations,

$$a_2 a_2^\dagger = \hat n(\hat n - 1), \quad a_2^\dagger a_2 = (\hat n + 1)(\hat n + 2).$$

Similarly, we have relations among $(\hat n, a_3^\dagger := {a^\dagger}^3, a_3 := a^3, a^\dagger, a, 1)$.

$$[a_3, a_3^\dagger] = 9\hat n^2 + 9\hat n + 6, \quad [\hat n, a_3^\dagger] = 3a_3^\dagger, \quad [\hat n, a_3] = -3a_3,$$

$$a_3 a_3^\dagger = \hat n(\hat n - 1)(\hat n - 2), \quad a_3^\dagger a_3 = (\hat n + 1)(\hat n + 2)(\hat n + 3).$$

Such observations show us that the composite operator $a_p^\dagger := (a^\dagger)^p$ $(a_p := a^p)$ is natural. We call it $a_p^\dagger$ p-on when p is a prime number. For example, for $n = p \cdot m$, we have a relation, $|n\rangle = (a^\dagger)^n|0\rangle/\sqrt{n!} = (a_p^\dagger)^m|0\rangle/\sqrt{n!}$. This means that the individual monomial $(a^\dagger)^{n+m}$ should be regarded as a commutative ring, *i.e.*, $(a^\dagger)^n(a^\dagger)^m = (a^\dagger)^m(a^\dagger)^n$. For later convenience, we also write $a_m^\dagger := (a^\dagger)^m$, $(a_m := a^m)$, and sometimes call it m-on though it may be, a little bit, overuse.

Then we have the following relations;

Proposition 2.1.

$$a_\ell a_\ell^\dagger = (\hat n{+}1)(\hat n{+}2)\dots(\hat n{+}\ell), \quad a_\ell^\dagger a_\ell = \hat n(\hat n{-}1)\dots(\hat n{-}\ell{+}1), \quad [\hat n, a_\ell^\dagger] = \ell a_\ell^\dagger, \quad [\hat n, a_\ell] = -\ell a_\ell.$$

Proof. They are proved by the induction. We have $\ell = 1, 2, 3$ cases. Let us show $[a, {a^\dagger}^\ell] = \ell {a^\dagger}^{\ell-1}$ because

$$[a, a_\ell^\dagger] = [a, {a^\dagger}^\ell] = a^\dagger[a, {a^\dagger}^{\ell-1}] + [a, a^\dagger]{a^\dagger}^{\ell-1}.$$

Thus $[\hat n, {a^\dagger}^\ell]$ is computed. The first formula is obtained by

$$\begin{aligned} a_\ell a_\ell^\dagger &= a^\ell {a^\dagger}^\ell \\ &= a^{\ell-1}(a{a^\dagger}^{\ell-1})a^\dagger \\ &= a^{\ell-1}({a^\dagger}^{\ell-1}a + (\ell-1){a^\dagger}^{\ell-1})a^\dagger \\ &= a^{\ell-1}{a^\dagger}^{\ell-1}(aa^\dagger + (\ell-1)) \\ &= a^{\ell-1}{a^\dagger}^{\ell-1}(\hat n + \ell). \end{aligned}$$

Similarly we have the relations of a_ℓ. □

Let us consider the relations to the Riemann ζ function. As formal expressions, we have

$$\mathrm{e}^{a^\dagger} - 1 = a^\dagger + \frac{1}{2!}(a^\dagger)^2 + \frac{1}{3!}(a^\dagger)^3 + \frac{1}{4!}(a^\dagger)^4 \cdots, \quad \frac{a}{1-a} = a + (a)^2 + (a)^3 + (a)^4 \cdots.$$

By letting (see the Appendix),

$$\langle n|(a^\dagger a)^{-s}|m\rangle := \int_0^\infty \frac{d\beta}{\beta}\beta^s \langle n|\mathrm{e}^{-\beta(a^\dagger a)}|m\rangle,$$

the following proposition holds:

Proposition 2.2. *The Riemann ζ function is expressed by*

$$\zeta(s) = \langle 0|\frac{a}{1-a}(a^\dagger a)^{-s}(\mathrm{e}^{a^\dagger}-1)|0\rangle.$$

Proof. Due to the independence of each state, we have

$$\langle 0|a^\ell (a^\dagger a)^{-s}(a^\dagger)^m|0\rangle = m!\delta_{n,m}m^{-s},$$

and then the relation is obtained. □

Let $\wp$ be the set of the prime numbers. Using p-on, the Euler product expression (1.2) is expressed by the following proposition;

Proposition 2.3. (Euler product expression)

$$\zeta(s) = \prod_{p\in\wp}\zeta_p(s), \quad \zeta_p(s) = \frac{1}{p!}\langle 0|\left(a_p\frac{1}{1-(a^\dagger a)^{-s}}a_p^\dagger\right)|0\rangle.$$

Proof. For a prime number p, we have $\langle 0|a_p(a^\dagger a)^{-\ell s}a_p^\dagger|0\rangle = p!p^{-ms}$ and

$$\begin{aligned}\langle p|\frac{1}{1-(a^\dagger a)^{-s}}|p\rangle &= \int_0^\infty \frac{d\beta}{\beta}\frac{1}{1-\beta^s}\langle p|\mathrm{e}^{-\beta(a^\dagger a)}|p\rangle \\ &= \int_0^\infty \frac{d\beta}{\beta}(1+\beta^s+\beta^{2s}+\beta^{3s}+\cdots)\langle p|\mathrm{e}^{-\beta(a^\dagger a)}|p\rangle.\end{aligned}$$

□

3. Quantum p-on and quantum Euler product expression

In this section, I will propose the quantum p-on and the quantum Euler product expression along the line of the concept of p-on in the previous section. Let us define operators A_m and $A_m^\dagger$ which are elements of endmorphisms $\mathfrak{a}_+$ and $\mathfrak{a}_-$, *i.e.*,

$$A_m^\dagger : \mathfrak{a}_+ \to \mathfrak{a}_+, \qquad A_m : \mathfrak{a}_- \to \mathfrak{a}_-,$$

by

$$A_n^\dagger \cdot (a^\dagger)^m|0\rangle = \frac{m!}{(mn)!}(a^\dagger)^{mn}|0\rangle, \quad \langle 0|a_m \cdot A_n = \langle 0|a^{mn}.$$

Physically speaking, $A_m^\dagger$ is the creation operator which creates m ℓ-ons when it acts on $a_\ell^\dagger|0\rangle$.

From the definition, we have their multiplicity;

Lemma 3.1.

$$A_m^{\dagger\,n} = A^\dagger_{m^n}, \quad A_m^n = A_{m^n}, \quad A^\dagger_m A^\dagger_n = A^\dagger_n A^\dagger_m = A^\dagger_{nm}, \quad A_m A_n = A_n A_m = A_{mn}.$$

Proof. $A_m^{\dagger\,2}((a^\dagger)^\ell)|0\rangle = \frac{\ell!}{(m\ell)!}A^\dagger_m((a^\dagger)^{m\ell})|0\rangle = \frac{\ell!}{(m\ell)!}\frac{(m\ell)!}{(m^2\ell)!}((a^\dagger)^{m^2\ell})|0\rangle = A^\dagger{}_{m^2}((a^\dagger)^\ell)|0\rangle.$ □

Thus we have the proposition:

Proposition 3.1. *$\mathfrak{A}^+ := \mathbb{C}[[\{A^\dagger_p\}_{p\in\wp}]]$ are $\mathfrak{A}^- := \mathbb{C}[[\{A_p\}_{p\in\wp}]]$ are commutative rings.*

Further we have their properties.

Lemma 3.2.

$$\left(\prod_{p\in\wp}\frac{1}{1-A^\dagger_p}\right)a^\dagger|0\rangle = \sum_{n=1}^\infty \frac{1}{n!}|n\rangle, \quad \prod_{p\in\wp}\left(1-A^\dagger_p\right)\sum_{n=1}^\infty\frac{1}{n!}|n\rangle = |1\rangle,$$

$$\langle 0|a\left(\prod_{p\in\wp}\frac{1}{1-A_p}\right) = \sum_{n=1}^\infty\langle n|, \quad \sum_{n=1}^\infty\langle n|\prod_{p\in\wp}(1-A_p) = \langle 1|.$$

Proof. Noting their commutativity,

$$\left(\frac{1}{1-A^\dagger_p}\right) = 1 + A^\dagger_p + A_p^{\dagger\,2} + A_p^{\dagger\,3}\cdots = 1 + A^\dagger_p + A^\dagger_{p^2} + A^\dagger_{p^3}\cdots.$$

Since every integer n is uniquely given by $n = \prod_{i=1}^{\ell_n} p_i^{r_i}$ for certain prime numbers p_i and positive numbers r_i $(i = 1, \cdots, \ell_n)$, we have

$$\prod_{p\in\wp}\left(1 + A^\dagger_p + A_p^{\dagger\,2} + A_p^{\dagger\,3}\cdots\right)a^\dagger|0\rangle = \sum_{n=1}|n\rangle.$$

On the other hand, we have

$$\left(1-A^\dagger_p\right)\left(\frac{1}{1-A^\dagger_p}\right) = 1 + A^\dagger_p + A_p^{\dagger\,2} + A_p^{\dagger\,3}\cdots - \left(A^\dagger_p + A_p^{\dagger\,2} + A_p^{\dagger\,3}\cdots\right) = 1.$$

□

Hence we have an quantum version of the Euler product expression (1.2):

Proposition 3.2. (quantum Euler product expression)

$$\zeta(s) = \langle 0|a\left(\prod_{q\in\wp}\frac{1}{1-A_q}\right)(a^\dagger a)^{-s}\left(\prod_{p\in\wp}\frac{1}{1-A^\dagger_p}\right)a^\dagger|0\rangle.$$

Proof. From the definition and Lemma 3.2, we have the relations

$$(\mathrm{e}^{a^\dagger}-1)|0\rangle = \left(\prod_{p\in\wp}(1 + A^\dagger_p + A^\dagger_{p^2} + \cdots)\right)a^\dagger|0\rangle,$$

and

$$\langle 0|\frac{a}{1-a} = \langle 0|a\left(\prod_{p\in\wp}(1+A_p+A_{p^2}+\cdots)\right).$$

Due to the above expression, $\zeta(s)$ is equal to

$$\langle 0|a\left(\prod_{p\in\wp}(1+A_p+A_{p^2}+\cdots)\right)(a^\dagger a)^{-s}\left(\prod_{p\in\wp}(1+A_p^\dagger+A_{p^2}^\dagger+\cdots)\right)a^\dagger|0\rangle.$$

The independence of each p-on gives the relation. $\square$

Noting

$$(1-A_p^\dagger)\left(\prod_{q\in\wp}\frac{1}{1-A_q^\dagger}\right)a^\dagger|0\rangle = \left(\prod_{q\in\wp,q\neq p}\frac{1}{1-A_q^\dagger}\right)a^\dagger|0\rangle,$$

the ζ function might be decomposed to the ζ_p function. Further due to interesting relation,

$$\frac{1}{m^\ell!}\langle 0|a_{m^\ell}(aa^\dagger)^{-s}a_{m^\ell}^\dagger|0\rangle = \frac{1}{m!}\langle 0|a_m(aa^\dagger)^{-s\ell}a_m^\dagger|0\rangle$$

we have the Proposition;

Proposition 3.3. (quantum Euler product expression II)

$$\zeta_p(s) = \langle 0|a\left(\frac{1}{1-A_p}\right)(a^\dagger a)^{-s}\left(\frac{1}{1-A_p^\dagger}\right)a^\dagger|0\rangle$$

Proof.

$$\langle 0|a\left(\frac{1}{1-A_p}\right)(a^\dagger a)^{-s}\left(\frac{1}{1-A_p^\dagger}\right)a^\dagger|0\rangle = \langle 0|a_p\frac{1}{1-(a^\dagger a)^{-s}}a_p^\dagger|0\rangle = \zeta_p(s).$$

$\square$

The above relation means

$$\begin{aligned}\zeta_p(s) &= \langle 0|a\,(1+A_p+A_{p^2}+A_{p^3}+\cdots)\,(a^\dagger a)^{-s}\left(1+A_p^\dagger+A_{p^2}^\dagger+A_{p^3}^\dagger\cdots\right)a^\dagger|0\rangle \\ &= \langle 0|\,(a+a_p+a_{p^2}+a_{p^3}+\cdots)\,(a^\dagger a)^{-s}\left(a_p^\dagger+a_{p^2}^\dagger+a_{p^3}^\dagger\cdots\right)|0\rangle\end{aligned} \tag{3.1}$$

4. Discussion

First we comments on an identification the quartet

$$(\mathbb{C}[[a, a^\dagger]], \mathfrak{a}^-, \mathfrak{a}^+, \langle 0|\mathbb{C}[[a, a^\dagger]]|0\rangle),$$

as

$$(\mathbb{C}[[\frac{d}{dz}, z]], \mathbb{C}[[\frac{d}{dz}]], \mathbb{C}[[z]], \frac{1}{2\pi\sqrt{-1}}\oint \frac{dz}{z}\mathbb{C}[[\frac{d}{dz}, z]]\cdot 1)$$

for $z \in \mathbb{C}P^1$. This identification could be regarded as a transformation between harmonic oscillator and operators on the Fourier series. We should note that $[a_2, a_2^\dagger]$ is regarded as $\left[\frac{d^2}{dz^2}, z^2\right] = 4x\frac{d}{dx} + 2$, which may be related to the quadratic differentials on Riemann surfaces [FK, Chapter VII.2]. Further instead of $\mathbb{C}P^1$, for example in [MP] for a algebraic curve, *e.g.*, $y^r = x^s + \lambda_{s-1}x^{s-1} + \cdots + \lambda_0$, at its infinite point, the local parameter z_∞ behaves like

$$z_\infty^r = 1/x + O(z_\infty^{r+1}), \quad z_\infty^s = 1/y + O(z_\infty^{r+1}).$$

In other words, $1/x$ and $1/y$ behave like $a_r^\dagger$ and $a_s^\dagger$ respectively. When we consider more general algebraic curves, there naturally appear relations among $(a_{\ell_1}, a_{\ell_2}, \cdots, a_{\ell_k})$. They are related to nonlinear integrable system and several physical phenomenon [BBEIM]. The dynamics of $\{z^\ell\}$ in the orthogonal polynomial is connected with the integrable system and the random matrix problem [S]; the random matrix is also connected with ζ function [Me]. Thus this interpretation is not trivial and is very natural from the viewpoint.

Further we note that for the system $(x, d/dx)$, $A^\dagger$ could be regarded as

$$\frac{d^\ell}{dx^\ell}A_m = \frac{d^{\ell m}}{dx^{\ell m}}, \quad A_m^\dagger x^\ell = \frac{\ell!}{(m\ell)!}x^{m\ell}.$$

Thus the p-on picture is natural from the viewpoint of the the identification.

Secondly we give some comments on Proposition 3.2 and Proposition 3.3 of quantum Euler product expression. The quantum Euler product expression in Proposition 3.3 is reduced to the ordinary Euler product expression (1.2). In other words, in the harmonic oscillator problem, the natural commutative ring structure exists and provides the relations to the Euler product expression (1.2) of the Riemann ζ function. I have a quantum mechanical interpretation of the Euler product expression and its quantum meaning as a relation to p-on.

I should emphasize that even behind the Planck black body problem and the Casimir effect, these expressions exist. (3.1) shows that there exist excitations of p-on, p^2-on, p^3-on and so on for each prime number. The multi-photon absorption in quantum optics is a sure sign of these excitations. It implies one of answers why the quantum mechanics provides a connection with number theory and p-adic structure [M, M1, M2, MO, V1, V2, VVZ].

Further from the definition, we may have the relation

$$A_m \cdot (a^\dagger)^n|0\rangle = \begin{cases} \frac{n!}{(n/m)!}(a^\dagger)^{n/m}|0\rangle & \text{if } m|n, \\ 0 & \text{otherwise .} \end{cases}$$

In other words, for a prime number p, we have

$$A_p \cdot (a^\dagger)^n|0\rangle = \begin{cases} \frac{(p^\ell m)!}{(p^{\ell-1}m)!}(a^\dagger)^{mp^{\ell-1}}|0\rangle & \text{if } n = p^\ell m, \ (\ell \ge 1, p \not| m), \\ 0 & \text{otherwise .} \end{cases}$$

This reminds me of the absolute derivation [KOW],

$$\frac{\partial}{\partial p} : \mathbb{Z} \to \mathbb{Z},$$

$$\frac{\partial}{\partial p} n = \begin{cases} \ell p^{\ell-1} m & \text{if } n = p^\ell m, \ (\ell \ge 1, p \not| m), \\ 0 & \text{otherwise ,} \end{cases}$$

for a prime number p, which is introduced for the study of the Riemann ζ function.

I believe that this commutative ring structure in the harmonic oscillator is quite interesting and gives an answer of the question why number theory is connected with quantum mechanics and now we interpret the Euler product expression in the harmonic oscillator. It should be emphasized that even behind the Planck black body problem and the Casimir effect, the commutative ring structure and p^ℓ-on exist.

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5. Appendix L-function

In this Appendix, I will show the interpretation L-functions and the Riemann ζ function in number theory from a statistical mechanical viewpoint. In statistical mechanics, we consider partition functions which are generators of expectations in canonical ensembles. The partition function for a statistical mechanical system A is defined by

$$Z[\beta] := \sum_{\text{all of states } s \text{ in } A} \mathrm{e}^{-\beta E(s)}, \tag{5.1}$$

where $1/\beta$ is a temperature of the system A and $E(s)$ is an energy of a state $s \in A$. By using spectral decomposition, we can also express it as

$$Z[\beta] = \sum_E \sum_{s \in S_E} \mathrm{e}^{-\beta E}, \tag{5.2}$$

where S_E is a subset of A which has energy E. Further we sometimes rewrite this

$$Z[\beta] = \sum_E c_E \mathrm{e}^{-\beta E}, \tag{5.3}$$

where c_E is number of S_E, *i.e.*, $c_E := \#S_E$. (5.3) is called the energy-representation.

Here we note that S_E is equivalent with respect to the energy E. The equivalence means that there might exist a group G_E which simply transitively acts on S_E. Then the c_E must be an invariance of G_E or a group ring $R = \mathbb{Z}[G_E]$. For example, c_E and S_E might be related to an identity representation,

$$\hat{c}_E := \sum_{x \in G_E} x.$$

In fact for a map $\varphi : \mathbb{Z}[G_E] \ni \sum a_i x_i \to \sum a_i \in \mathbb{Z}$, $\varphi(\hat{c}_E) = c_E$.

In statistics, we sometimes deal with sequence of the n-th order expectation value (n-th moment) instead of the generator itself. Thus it is natural to introduce an n-th moment,

$$\begin{aligned} K[s] &:= \int_0^\infty \beta^s Z[\beta] \frac{d\beta}{\beta} \\ &= \Gamma(s) \sum_E \frac{c_E}{E^{-s}}, \end{aligned} \tag{5.4}$$

where s is a natural number. (Here we note that existence of the integral (5.4) is asserted by, for example, (3.9) Theorem in [Du].) It is remarked that s is sometimes extended to a complex number by analytical continuity. This $K[s]$ is called the generalized ζ-function which appears in [EORBZ, M0].

Let E be parameterized by an integer or $E = n \in \mathbb{Z}$. When $c_n = 1$, we have the Riemann ζ function [Pa],

$$K[s] = \sum_{n=1} \frac{1}{n^s} = \zeta(s),$$

which is the main theme of this article.

Let c_n satisfy

$$c_n = \begin{cases} 2 & \text{for } n \equiv 1, 7 \bmod 8, \\ 0 & \text{for } n \equiv 3, 5 \bmod 8, \\ 1 & \text{for otherwise} . \end{cases}$$

$$K[s] = L(s, \chi) + \zeta(s),$$

where $L(s, \chi)$ is the Dirichlet characteristics which is given by [IR],

$$L(s, \chi) = \sum_{n=1} \frac{\chi(n)}{n^s},$$

and

$$\chi(n) := \begin{cases} 1 & \text{for } n \equiv 1, 7 \bmod 8, \\ -1 & \text{for } n \equiv 3, 5 \bmod 8, \\ 0 & \text{for otherwise} . \end{cases}$$

Further (5.3) is also related to the Gauss sum [IR, BK, MO] if $\begin{pmatrix} n \\ p \end{pmatrix}$ and $\beta = \dfrac{\sqrt{-1}}{p}$.

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Shigeki Matsutani
e-mail:RXB01142nifty.com
8-21-1 Higashi-Linkan
Sagamihara 228-0811 Japan